Game Theory*

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*This is the draft of an introductory survey of game theory, prepared for the *Encyclopedia of Information Systems*, Academic Press, to appear in 2002.

Glossary

Backward induction

Backward induction is a technique to solve a game of perfect information. It first considers the moves that are the last in the game, and determines the best move for the player in each case. Then, taking these as given future actions, it proceeds backwards in time, again determining the best move for the respective player, until the beginning of the game is reached.

Common knowledge

A fact is common knowledge if all players know it, and know that they all know it, and so on. The structure of the game is often assumed to be common knowledge among the players.

Dominating strategy

A strategy dominates another strategy of a player if it always gives a better payoff to that player, regardless of what the other players are doing. It weakly dominates the other strategy if it is always at least as good.

Extensive game

An extensive game (or extensive form game) describes with a tree how a game is played. It depicts the order in which players make moves, and the information each player has at each decision point.

Game

A game is a formal description of a strategic situation.

Game theory

Game theory is the formal study of decision-making where several players must make choices that potentially affect the interests of the other players.

Mixed strategy

A mixed strategy is an active randomization, with given probabilities, that determines the player's decision. As a special case, a mixed strategy can be the deterministic choice of one of the given pure strategies.

Nash equilibrium

A Nash equilibrium, also called strategic equilibrium, is a list of strategies, one for each player, which has the property that no player can unilaterally change his strategy and get a better payoff.

Payoff

A payoff is a number, also called utility, that reflects the desirability of an outcome to a player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities. The expected payoff incorporates the player's attitude towards risk.

Perfect information

A game has perfect information when at any point in time only one player makes a move, and knows all the actions that have been made until then.

Player

A player is an agent who makes decisions in a game.

Rationality

A player is said to be rational if he seeks to play in a manner which maximizes his own payoff. It is often assumed that the rationality of all players is common knowledge.

Strategic form

A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies. The resulting payoffs are presented in a table with a cell for each strategy combination.

Strategy

In a game in strategic form, a strategy is one of the given possible actions of a player. In an extensive game, a strategy is a complete plan of choices, one for each decision point of the player.

Zero-sum game

A game is said to be zero-sum if for any outcome, the sum of the payoffs to all players is zero. In a two-player zero-sum game, one player's gain is the other player's loss, so their interests are diametrically opposed.

1 What is game theory?

Game theory is the formal study of conflict and cooperation. Game theoretic concepts apply whenever the actions of several agents are interdependent. These agents may be individuals, groups, firms, or any combination of these. The concepts of game theory provide a language to formulate, structure, analyze, and understand strategic scenarios.

History and impact of game theory

The earliest example of a formal game-theoretic analysis is the study of a duopoly by Antoine Cournot in 1838. The mathematician Emile Borel suggested a formal theory of games in 1921, which was furthered by the mathematician John von Neumann in 1928 in a "theory of parlor games." Game theory was established as a field in its own right after the 1944 publication of the monumental volume *Theory of Games and Economic Behavior* by von Neumann and the economist Oskar Morgenstern. This book provided much of the basic terminology and problem setup that is still in use today.

In 1950, John Nash demonstrated that finite games have always have an equilibrium point, at which all players choose actions which are best for them given their opponents' choices. This central concept of noncooperative game theory has been a focal point of analysis since then. In the 1950s and 1960s, game theory was broadened theoretically and applied to problems of war and politics. Since the 1970s, it has driven a revolution in economic theory. Additionally, it has found applications in sociology and psychology, and established links with evolution and biology. Game theory received special attention in 1994 with the awarding of the Nobel prize in economics to Nash, John Harsanyi, and Reinhard Selten.

At the end of the 1990s, a high-profile application of game theory has been the design of auctions. Prominent game theorists have been involved in the design of auctions for allocating rights to the use of bands of the electromagnetic spectrum to the mobile telecommunications industry. Most of these auctions were designed with the goal of allocating these resources more efficiently than traditional governmental practices, and additionally raised billions of dollars in the United States and Europe.

Game theory and information systems

The internal consistency and mathematical foundations of game theory make it a prime tool for modeling and designing automated decision-making processes in interactive environments. For example, one might like to have efficient bidding rules for an auction website, or tamper-proof automated negotiations for purchasing communication bandwidth. Research in these applications of game theory is the topic of recent conference and journal papers (see, for example, Binmore and Vulkan, "Applying game theory to automated negotiation," *Netnomics* Vol. 1, 1999, pages 1–9) but is still in a nascent stage. The automation of strategic choices enhances the need for these choices to be made efficiently, and to be robust against abuse. Game theory addresses these requirements.

As a mathematical tool for the decision-maker the strength of game theory is the methodology it provides for structuring and analyzing problems of strategic choice. The process of formally modeling a situation as a game requires the decision-maker to enumerate explicitly the players and their strategic options, and to consider their preferences and reactions. The discipline involved in constructing such a model already has the potential of providing the decision-maker with a clearer and broader view of the situation. This is a "prescriptive" application of game theory, with the goal of improved strategic decision making. With this perspective in mind, this article explains basic principles of game theory, as an introduction to an interested reader without a background in economics.

2 Definitions of games

The object of study in game theory is the *game*, which is a formal model of an interactive situation. It typically involves several *players*; a game with only one player is usually called a *decision problem*. The formal definition lays out the players, their preferences, their information, the strategic actions available to them, and how these influence the outcome.

Games can be described formally at various levels of detail. A *coalitional* (or *cooperative*) game is a high-level description, specifying only what payoffs each potential group, or coalition, can obtain by the cooperation of its members. What is not made explicit is the process by which the coalition forms. As an example, the players may be several parties in parliament. Each party has a different strength, based upon the number of seats occupied by party members. The game describes which coalitions of parties can form a majority, but does not delineate, for example, the negotiation process through which an agreement to vote en bloc is achieved.

Cooperative game theory investigates such coalitional games with respect to the relative amounts of power held by various players, or how a successful coalition should divide its proceeds. This is most naturally applied to situations arising in political science or international relations, where concepts like power are most important. For example, Nash proposed a solution for the division of gains from agreement in a bargaining problem which depends solely on the relative strengths of the two parties' bargaining position. The amount of power a side has is determined by the usually inefficient outcome that results when negotiations break down. Nash's model fits within the cooperative framework in that it does not delineate a specific timeline of offers and counteroffers, but rather focuses solely on the outcome of the bargaining process.

In contrast, *noncooperative game theory* is concerned with the analysis of strategic choices. The paradigm of noncooperative game theory is that the details of the ordering and timing of players' choices are crucial to determining the outcome of a game. In contrast to Nash's cooperative model, a noncooperative model of bargaining would posit a specific process in which it is prespecified who gets to make an offer at a given time. The term "noncooperative" means this branch of game theory explicitly models the process of players making choices out of their own interest. Cooperation can, and often does, arise in noncooperative models of games, when players find it in their own best interests.

Branches of game theory also differ in their assumptions. A central assumption in many variants of game theory is that the players are *rational*. A rational player is one who always chooses an action which gives the outcome he most prefers, given what he expects his opponents to do. The goal of game-theoretic analysis in these branches, then, is to predict how the game will be played by rational players, or, relatedly, to give advice on how best to play the game against opponents who are rational. This rationality assumption can be relaxed, and the resulting models have been more recently applied to the analysis of observed behavior (see Kagel and Roth, eds., *Handbook of Experimental Economics*, Princeton Univ. Press, 1997). This kind of game theory can be viewed as more "descriptive" than the prescriptive approach taken here.

This article focuses principally on noncooperative game theory with rational players. In addition to providing an important baseline case in economic theory, this case is designed so that it gives good advice to the decision-maker, even when – or perhaps especially when – one's opponents also employ it.

Strategic and extensive form games

The *strategic form* (also called *normal form*) is the basic type of game studied in noncooperative game theory. A game in strategic form lists each player's strategies, and the outcomes that result from each possible combination of choices. An outcome is represented by a separate *payoff* for each player, which is a number (also called *utility*) that measures how much the player likes the outcome.

The *extensive form*, also called a *game tree*, is more detailed than the strategic form of a game. It is a complete description of how the game is played over time. This includes the order in which players take actions, the information that players have at the time they must take those actions, and the times at which any uncertainty in the situation is resolved. A game in extensive form may be analyzed directly, or can be converted into an equivalent strategic form.

Examples in the following sections will illustrate in detail the interpretation and analysis of games in strategic and extensive form.

3 Dominance

Since all players are assumed to be rational, they make choices which result in the outcome they prefer most, given what their opponents do. In the extreme case, a player may have two strategies A and B so that, given any combination of strategies of the other players, the outcome resulting from A is better than the outcome resulting from B. Then strategy A is said to *dominate* strategy B. A rational player will never choose to play a dominated strategy. In some games, examination of which strategies are dominated results in the conclusion that rational players could only ever choose one of their strategies. The following examples illustrate this idea.

Example: Prisoner's Dilemma

The Prisoner's Dilemma is a game in strategic form between two players. Each player has two strategies, called "cooperate" and "defect," which are labeled C and D for player I and c and d for player II, respectively. (For simpler identification, upper case letters are used for strategies of player I and lower case letters for player II.)

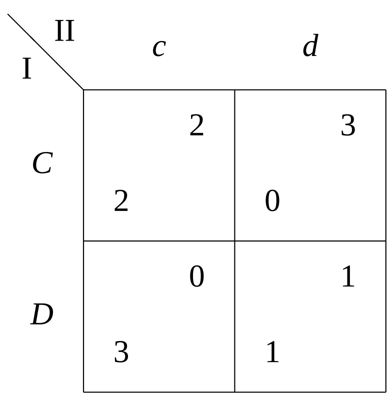

Figure 1. The Prisoner's Dilemma game.

Figure 1 shows the resulting payoffs in this game. Player I chooses a row, either C or D, and simultaneously player II chooses one of the columns c or d. The strategy combination (C, c) has payoff 2 for each player, and the combination (D, d) gives each player payoff 1. The combination (C, d) results in payoff 0 for player I and 3 for player II, and when (D, c) is played, player I gets 3 and player II gets 0.

Any two-player game in strategic form can be described by a table like the one in Figure 1, with rows representing the strategies of player I and columns those of player II. (A player may have more than two strategies.) Each strategy combination defines a payoff pair, like $(3, 0)$ for (D, c), which is given in the respective table entry. Each cell of the table shows the payoff to player I at the (lower) left, and the payoff to player II at the (right) top. These staggered payoffs, due to Thomas Schelling, also make transparent when, as here, the game is symmetric between the two players. Symmetry means that the game stays the same when the players are exchanged, corresponding to a reflection along the diagonal shown as a dotted line in Figure 2. Note that in the strategic form, there is no order between player I and II since they act simultaneously (that is, without knowing the other's action), which makes the symmetry possible.

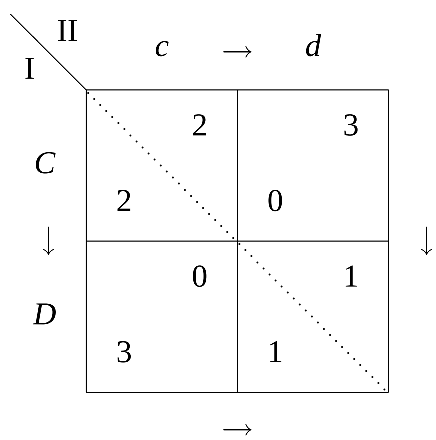

Figure 2. The game of Figure 1 with annotations, implied by the payoff structure. The dotted line shows the symmetry of the game. The arrows at the left and right point to the preferred strategy of player I when player II plays the left or right column, respectively. Similarly, the arrows at the top and bottom point to the preferred strategy of player II when player I plays top or bottom.

In the Prisoner's Dilemma game, "defect" is a strategy that dominates "cooperate." Strategy D of player I dominates C since if player II chooses c, then player I's payoff is 3 when choosing D and 2 when choosing C; if player II chooses d, then player I receives 1 for D as opposed to 0 for C. These preferences of player I are indicated by the downward-pointing arrows in Figure 2. Hence, D is indeed always better and dominates C. In the same way, strategy d dominates c for player II.

No rational player will choose a dominated strategy since the player will always be better off when changing to the strategy that dominates it. The unique outcome in this game, as recommended to utility-maximizing players, is therefore (D, d) with payoffs $(1, 1)$. Somewhat paradoxically, this is less than the payoff $(2, 2)$ that would be achieved when the players chose (C, c).

The story behind the name "Prisoner's Dilemma" is that of two prisoners held suspect of a serious crime. There is no judicial evidence for this crime except if one of the prisoners testifies against the other. If one of them testifies, he will be rewarded with immunity from prosecution (payoff 3), whereas the other will serve a long prison sentence (payoff 0). If both testify, their punishment will be less severe (payoff 1 for each). However, if they both "cooperate" with each other by not testifying at all, they will only be imprisoned briefly, for example for illegal weapons possession (payoff 2 for each). The "defection" from that mutually beneficial outcome is to testify, which gives a higher payoff no matter what the other prisoner does, with a resulting lower payoff to both. This constitutes their "dilemma."

Prisoner's Dilemma games arise in various contexts where individual "defections" at the expense of others lead to overall less desirable outcomes. Examples include arms races, litigation instead of settlement, environmental pollution, or cut-price marketing, where the resulting outcome is detrimental for the players. Its game-theoretic justification on individual grounds is sometimes taken as a case for treaties and laws, which enforce cooperation.

Game theorists have tried to tackle the obvious "inefficiency" of the outcome of the Prisoner's Dilemma game. For example, the game is fundamentally changed by playing it more than once. In such a *repeated game*, patterns of cooperation can be established as rational behavior when players' fear of punishment in the future outweighs their gain from defecting today.

Example: Quality choice

The next example of a game illustrates how the principle of elimination of dominated strategies may be applied iteratively. Suppose player I is an internet service provider and player II a potential customer. They consider entering into a contract of service provision for a period of time. The provider can, for himself, decide between two levels of quality of service, *High* or *Low*. High-quality service is more costly to provide, and some of the cost is independent of whether the contract is signed or not. The level of service cannot be put verifiably into the contract. High-quality service is more valuable than low-quality service to the customer, in fact so much so that the customer would prefer not to buy the service if she knew that the quality was low. Her choices are to *buy* or *not to buy* the service.

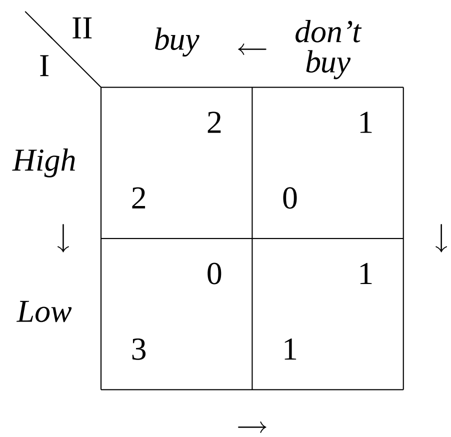

Figure 3. High-low quality game between a service provider (player I) and a customer (player II).

Figure 3 gives possible payoffs that describe this situation. The customer prefers to buy if player I provides high-quality service, and not to buy otherwise. Regardless of whether the customer chooses to buy or not, the provider always prefers to provide the low-quality service. Therefore, the strategy *Low* dominates the strategy *High* for player I.

Now, since player II believes player I is rational, she realizes that player I always prefers *Low*, and so she anticipates low quality service as the provider's choice. Then she prefers *not to buy* (giving her a payoff of 1) to *buy* (payoff 0). Therefore, the rationality of both players leads to the conclusion that the provider will implement low-quality service and, as a result, the contract will not be signed.

This game is very similar to the Prisoner's Dilemma in Figure 1. In fact, it differs only by a single payoff, namely payoff 1 (rather than 3) to player II in the top right cell in the table. This reverses the top arrow from right to left, and makes the preference of player II dependent on the action of player I. (The game is also no longer symmetric.) Player II does not have a dominating strategy. However, player I still does, so that the resulting outcome, seen from the "flow of arrows" in Figure 3, is still unique. Another way of obtaining this outcome is the successive elimination of dominated strategies: First, *High* is eliminated, and in the resulting smaller game where player I has only the single strategy *Low* available, player II's strategy *buy* is dominated and also removed.

As in the Prisoner's Dilemma, the individually rational outcome is worse for both players than another outcome, namely the strategy combination (*High*, *buy*) where high-quality service is provided and the customer signs the contract. However, that outcome is not credible, since the provider would be tempted to renege and provide only the low-quality service.

4 Nash equilibrium

In the previous examples, consideration of dominating strategies alone yielded precise advice to the players on how to play the game. In many games, however, there are no dominated strategies, and so these considerations are not enough to rule out any outcomes or to provide more specific advice on how to play the game.

The central concept of *Nash equilibrium* is much more general. A Nash equilibrium recommends a strategy to each player that the player cannot improve upon *unilaterally*, that is, given that the other players follow the recommendation. Since the other players are also rational, it is reasonable for each player to expect his opponents to follow the recommendation as well.

Example: Quality choice revisited

A game-theoretic analysis can highlight aspects of an interactive situation that could be changed to get a better outcome. In the quality game in Figure 3, for example, increasing the customer's utility of high-quality service has no effect unless the provider has an incentive to provide that service. So suppose that the game is changed by introducing an opt-out clause into the service contract. That is, the customer can discontinue subscribing to the service if she finds it of low quality.

The resulting game is shown in Figure 4. Here, low-quality service provision, even when the customer decides to buy, has the same low payoff 1 to the provider as when the

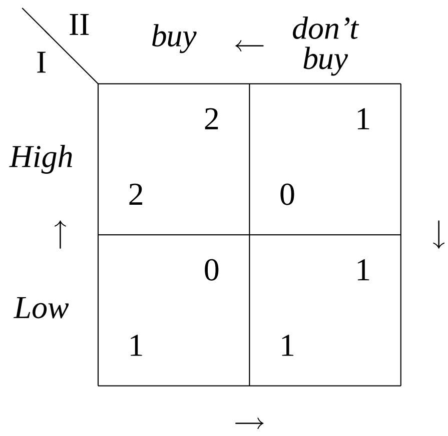

Figure 4. High-low quality game with opt-out clause for the customer. The left arrow shows that player I prefers *High* when player II chooses *buy*.

customer does not sign the contract in the first place, since the customer will opt out later. However, the customer still prefers not to buy when the service is *Low* in order to spare herself the hassle of entering the contract.

The changed payoff to player I means that the left arrow in Figure 4 points upwards. Note that, compared to Figure 3, only the provider's payoffs are changed. In a sense, the opt-out clause in the contract has the purpose of convincing the customer that the high-quality service provision is in the provider's own interest.

This game has no dominated strategy for either player. The arrows point in different directions. The game has *two* Nash equilibria in which each player chooses his strategy deterministically. One of them is, as before, the strategy combination (*Low, don't buy*). This is an equilibrium since *Low* is the *best response* (payoff-maximizing strategy) to *don't buy* and vice versa.

The second Nash equilibrium is the strategy combination (*High, buy*). It is an equilibrium since player I prefers to provide high-quality service when the customer buys, and conversely, player II prefers to buy when the quality is high. This equilibrium has a higher payoff to both players than the former one, and is a more desirable solution.

Both Nash equilibria are legitimate recommendations to the two players of how to play the game. Once the players have settled on strategies that form a Nash equilibrium, neither player has incentive to deviate, so that they will rationally stay with their strategies. This makes the Nash equilibrium a consistent solution concept for games. In contrast, a strategy combination that is *not* a Nash equilibrium is not a credible solution. Such a strategy combination would not be a reliable recommendation on how to play the game, since at least one player would rather ignore the advice and instead play another strategy to make himself better off.

As this example shows, a Nash equilibrium may be not unique. However, the previously discussed solutions to the Prisoner's Dilemma and to the quality choice game in Figure 3 are unique Nash equilibria. A dominated strategy can never be part of an equilibrium since a player intending to play a dominated strategy could switch to the dominating strategy and be better off. Thus, if elimination of dominated strategies leads to a unique strategy combination, then this is a Nash equilibrium. Larger games may also have unique equilibria that do not result from dominance considerations.

Equilibrium selection

If a game has more than one Nash equilibrium, a theory of strategic interaction should guide players towards the "most reasonable" equilibrium upon which they should focus. Indeed, a large number of papers in game theory have been concerned with "equilibrium refinements" that attempt to derive conditions that make one equilibrium more plausible or convincing than another. For example, it could be argued that an equilibrium that is better for both players, like (*High, buy*) in Figure 4, should be the one that is played.

However, the abstract theoretical considerations for equilibrium selection are often more sophisticated than the simple game-theoretical models they are applied to. It may be more illuminating to observe that a game has more than one equilibrium, and that this is a reason that players are sometimes stuck at an inferior outcome.

One and the same game may also have a different interpretation where a previously undesirable equilibrium becomes rather plausible. As an example, consider an alternative scenario for the game in Figure 4. Unlike the previous situation, it will have a symmetric description of the players, in line with the symmetry of the payoff structure.

Two firms want to invest in communication infrastructure. They intend to communicate frequently with each other using that infrastructure, but they decide independently on what to buy. Each firm can decide between *High* or *Low* bandwidth equipment (this time, the same strategy names will be used for both players). For player II, *High* and *Low* replace *buy* and *don't buy* in Figure 4. The rest of the game stays as it is.

The (unchanged) payoffs have the following interpretation for player I (which applies in the same way to player II by symmetry): A *Low* bandwidth connection works equally well (payoff 1) regardless of whether the other side has high or low bandwidth. However, switching from *Low* to *High* is preferable only if the other side has high bandwidth (payoff 2), otherwise it incurs unnecessary cost (payoff 0).

As in the quality game, the equilibrium (*Low*, *Low*) (the bottom right cell) is inferior to the other equilibrium, although in this interpretation it does not look quite as bad. Moreover, the strategy *Low* has obviously the better *worst-case* payoff, as considered for all possible strategies of the other player, no matter if these strategies are rational choices or not. The strategy *Low* is therefore also called a *max-min* strategy since it maximizes the minimum payoff the player can get in each case. In a sense, investing only in low-bandwidth equipment is a safe choice. Moreover, this strategy is part of an equilibrium, and entirely justified if the player expects the other player to do the same.

Evolutionary games

The bandwidth choice game can be given a different interpretation where it applies to a large *population* of identical players. Equilibrium can then be viewed as the outcome of a *dynamic process* rather than of conscious rational analysis.

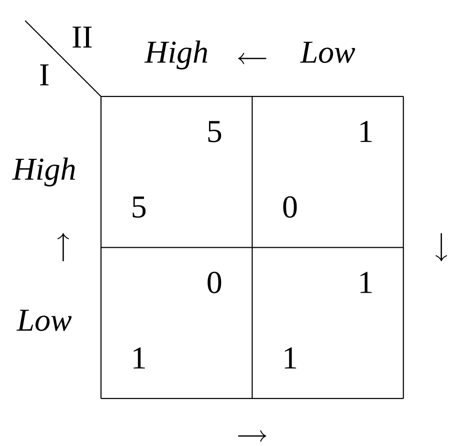

Figure 5. The bandwidth choice game.

Figure 5 shows the bandwidth choice game where each player has the two strategies *High* and *Low*. The positive payoff of 5 for each player for the strategy combination (*High, High*) makes this an even more preferable equilibrium than in the case discussed above.

In the evolutionary interpretation, there is a large population of individuals, each of which can adopt one of the strategies. The game describes the payoffs that result when two of these individuals meet. The dynamics of this game are based on assuming that each strategy is played by a certain *fraction* of individuals. Then, given this distribution of strategies, individuals with better *average payoff* will be more successful than others, so that their proportion in the population increases over time. This, in turn, may affect which strategies are better than others. In many cases, in particular in symmetric games with only two possible strategies, the dynamic process will move to an equilibrium.

In the example of Figure 5, a certain fraction of users connected to a network will already have *High* or *Low* bandwidth equipment. For example, suppose that one quarter of the users has chosen *High* and three quarters have chosen *Low*. It is useful to assign these as percentages to the columns, which represent the strategies of player II. A new user, as player I, is then to decide between *High* and *Low*, where his payoff depends on the given fractions. Here it will be $1/4 \times 5 + 3/4 \times 0 = 1.25$ when player I chooses *High*, and $1/4 \times 1 + 3/4 \times 1 = 1$ when player I chooses *Low*. Given the average payoff that player I can expect when interacting with other users, player I will be better off by choosing *High*, and so decides on that strategy. Then, player I joins the population as a *High* user. The proportion of individuals of type *High* therefore increases, and over time the advantage of that strategy will become even more pronounced. In addition, users replacing their equipment will make the same calculation, and therefore also switch from *Low* to *High*. Eventually, everyone plays *High* as the only surviving strategy, which corresponds to the equilibrium in the top left cell in Figure 5.

The long-term outcome where only high-bandwidth equipment is selected depends on there being an initial fraction of high-bandwidth users that is large enough. For example, if only ten percent have chosen *High*, then the expected payoff for *High* is $0.1 \times 5 + 0.9 \times 0 = 0.5$ which is less than the expected payoff 1 for *Low* (which is always 1, irrespective of the distribution of users in the population). Then, by the same logic as before, the fraction of *Low* users increases, moving to the bottom right cell of the game as the equilibrium. It is easy to see that the critical fraction of *High* users so that this will take off as the better strategy is 1/5. (When new technology makes high-bandwidth equipment cheaper, this increases the payoff 0 to the *High* user who is meeting *Low*, which changes the game.)

The evolutionary, population-dynamic view of games is useful because it does not require the assumption that all players are sophisticated and think the others are also rational, which is often unrealistic. Instead, the notion of rationality is replaced with the much weaker concept of *reproductive success*: strategies that are successful on average will be used more frequently and thus prevail in the end. This view originated in theoretical biology with Maynard Smith (*Evolution and the Theory of Games*, Cambridge University Press, 1982) and has since significantly increased in scope (see Hofbauer and Sigmund, *Evolutionary Games and Population Dynamics*, Cambridge University Press, 1998).

5 Mixed strategies

A game in strategic form does not always have a Nash equilibrium in which each player deterministically chooses one of his strategies. However, players may instead randomly select from among these *pure* strategies with certain probabilities. Randomizing one's own choice in this way is called a *mixed* strategy. Nash showed in 1951 that any finite strategic-form game has an equilibrium if mixed strategies are allowed. As before, an equilibrium is defined by a (possibly mixed) strategy for each player where no player can gain *on average* by unilateral deviation. Average (that is, *expected*) payoffs must be considered because the outcome of the game may be random.

Example: Compliance inspections

Suppose a consumer purchases a license for a software package, agreeing to certain restrictions on its use. The consumer has an incentive to violate these rules. The vendor would like to verify that the consumer is abiding by the agreement, but doing so requires inspections which are costly. If the vendor does inspect and catches the consumer cheating, the vendor can demand a large penalty payment for the noncompliance.

Figure 6 shows possible payoffs for such an inspection game. The standard outcome, defining the reference payoff zero to both vendor (player I) and consumer (player II),

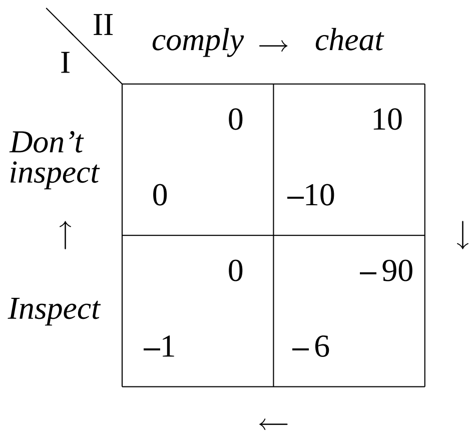

Figure 6. Inspection game between a software vendor (player I) and consumer (player II).

is that the vendor chooses *Don't inspect* and the consumer chooses to *comply*. Without inspection, the consumer prefers to *cheat* since that gives her payoff 10, with resulting negative payoff -10 to the vendor. The vendor may also decide to *Inspect*. If the consumer complies, inspection leaves her payoff 0 unchanged, while the vendor incurs a cost resulting in a negative payoff -1. If the consumer cheats, however, inspection will result in a heavy penalty (payoff -90 for player II) and still create a certain amount of hassle for player I (payoff -6).

In all cases, player I would strongly prefer if player II complied, but this is outside of player I's control. However, the vendor prefers to inspect if the consumer cheats (since -6 is better than -10), indicated by the downward arrow on the right in Figure 6. If the vendor always preferred *Don't inspect*, then this would be a dominating strategy and be part of a (unique) equilibrium where the consumer cheats.

The circular arrow structure in Figure 6 shows that this game has no equilibrium in pure strategies. If any of the players settles on a deterministic choice (like *Don't inspect* by player I), the best reponse of the other player would be unique (here *cheat* by player II), to which the original choice would *not* be a best reponse (player I prefers *Inspect* when the other player chooses *cheat*, against which player II in turn prefers to *comply*). The strategies in a Nash equilibrium must be best responses to each other, so in this game this fails to hold for any pure strategy combination.

Mixed equilibrium

What should the players do in the game of Figure 6? One possibility is that they prepare for the worst, that is, choose a *max-min* strategy. As explained before, a max-min strategy maximizes the player's worst payoff against all possible choices of the opponent. The max-min strategy for player I is to *Inspect* (where the vendor guarantees himself payoff -6), and for player II it is to *comply* (which guarantees her payoff 0). However, this is not a Nash equilibrium and hence not a stable recommendation to the two players, since player I could switch his strategy and improve his payoff.

A *mixed strategy* of player I in this game is to *Inspect* only with a certain probability. In the context of inspections, randomizing is also a practical approach that reduces costs. Even if an inspection is not certain, a sufficiently high chance of being caught should deter from cheating, at least to some extent.

The following considerations show how to find the probability of inspection that will lead to an equilibrium. If the probability of inspection is very low, for example one percent, then player II receives (irrespective of that probability) payoff 0 for *comply*, and payoff $0.99 \times 10 + 0.01 \times (-90) = 9$, which is bigger than zero, for *cheat*. Hence, player II will still cheat, just as in the absence of inspection.

If the probability of inspection is much higher, for example 0.2, then the expected payoff for *cheat* is $0.8 \times 10 + 0.2 \times (-90) = -10$, which is less than zero, so that player II prefers to *comply*. If the inspection probability is either too low or too high, then player II has a unique best response. As shown above, such a pure strategy cannot be part of an equilibrium.

Hence, the only case where player II herself could possibly randomize between her strategies is if both strategies give her the same payoff, that is, if she is *indifferent*. It is never optimal for a player to assign a positive probability to playing a strategy that is inferior, given what the other players are doing. It is not hard to see that player II is indifferent if and only if player I inspects with probability 0.1, since then the expected payoff for *cheat* is $0.9 \times 10 + 0.1 \times (-90) = 0$, which is then the same as the payoff for *comply*.

With this mixed strategy of player I (*Don't inspect* with probability 0.9 and *Inspect* with probability 0.1), player II is indifferent between her strategies. Hence, she can *mix* them (that is, play them randomly) without losing payoff. The only case where, in turn, the original mixed strategy of player I is a best response is if player I is indifferent. According to the payoffs in Figure 6, this requires player II to choose *comply* with probability 0.8 and *cheat* with probability 0.2. The expected payoffs to player I are then for *Don't inspect* $0.8 \times 0 + 0.2 \times (-10) = -2$, and for *Inspect* $0.8 \times (-1) + 0.2 \times (-6) = -2$, so that player I is indeed indifferent, and his mixed strategy is a best response to the mixed strategy of player II.

This defines the only Nash equilibrium of the game. It uses mixed strategies and is therefore called a *mixed* equilibrium. The resulting expected payoffs are -2 for player I and 0 for player II.

Interpretation of mixed strategy probabilities

The preceding analysis showed that the game in Figure 6 has a mixed equilibrium, where the players choose their pure strategies according to certain probabilities. These probabilities have several noteworthy features.

The equilibrium probability of 0.1 for *Inspect* makes player II indifferent between *comply* and *cheat*. This is based on the assumption that an *expected payoff* of 0 for *cheat*, namely $0.9 \times 10 + 0.1 \times (-90)$, is the same for player II as when getting the payoff 0 for certain, by choosing to *comply*. If the payoffs were monetary amounts (each payoff unit standing for one thousand dollars, say), one would not necessarily assume such a *risk neutrality* on the part of the consumer. In practice, decision-makers are typically *risk averse*, meaning they prefer the safe payoff of 0 to the gamble with an expectation of 0.

In a game-theoretic model with random outcomes (as in a mixed equilibrium), however, the payoff is not necessarily to be interpreted as money. Rather, the player's attitude towards risk is incorporated into the payoff figure as well. To take our example, the consumer faces a certain reward or punishment when cheating, depending on whether she is caught or not. Getting caught may not only involve financial loss but embarassment and other undesirable consequences. However, there is a certain probability of inspection (that is, of getting caught) where the consumer becomes indifferent between *comply* and *cheat*. If that probability is 1 against 9, then this indifference implies that the cost (negative payoff) for getting caught is 9 times as high as the reward for cheating successfully, as assumed by the payoffs in Figure 6. If the probability of indifference is 1 against

20, the payoff -90 in Figure 6 should be changed to -200. The units in which payoffs are measured are arbitrary. Like degrees on a temperature scale, they can be multiplied by a positive number and shifted by adding a constant, without altering the underlying preferences they represent.

In a sense, the payoffs in a game mimic a player's (consistent) willingness to bet when facing certain odds. With respect to the payoffs, which may distort the monetary amounts, players are then risk neutral. Such payoffs are also called *expected-utility* values. Expected-utility functions are also used in one-player games to model decisions under uncertainty.

The risk attitude of a player may not be known in practice. A game-theoretic analysis should be carried out for different choices of the payoff parameters in order to test how much they influence the results. Typically, these parameters represent the "political" features of a game-theoretic model, those most sensitive to subjective judgement, compared to the more "technical" part of a solution. In more involved inspection games, the technical part often concerns the optimal usage of limited inspection resources, whereas the political decision is when to raise an alarm and declare that the inspectee has cheated (see Avenhaus and Canty, *Compliance Quantified*, Cambridge University Press, 1996).

Secondly, mixing seems paradoxical when the player is indifferent in equilibrium. If player II, for example, can equally well *comply* or *cheat*, why should she gamble? In particular, she could *comply* and get payoff zero for certain, which is simpler and safer. The answer is that precisely because there is no incentive to choose one strategy over the other, a player can mix, and only in that case there can be an equilibrium. If player II would *comply* for certain, then the only optimal choice of player I is *Don't inspect*, making the choice of complying not optimal, so this is not an equilibrium.

The least intuitive aspect of mixed equilibrium is that the probabilities depend on the *opponent's payoffs* and not on the player's own payoffs (as long as the qualitative preference structure, represented by the arrows, remains intact). For example, one would expect that raising the penalty -90 in Figure 6 for being caught lowers the probability of cheating in equilibrium. In fact, it does not. What does change is the probability of inspection, which is reduced until the consumer is indifferent.

This dependence of mixed equilibrium probabilities on the opponent's payoffs can be explained terms of population dynamics. In that interpretation, Figure 6 represents an evolutionary game. Unlike Figure 5, it is a non-symmetric interaction between a vendor who chooses *Don't Inspect* and *Inspect* for certain fractions of a large number of interactions. Player II's actions *comply* and *cheat* are each chosen by a certain fraction of consumers involved in these interactions. If these fractions deviate from the equilibrium probabilities, then the strategies that do better will increase. For example, if player I chooses *Inspect* too often (relative to the penalty for a cheater who is caught), the fraction of cheaters will decrease, which in turn makes *Don't Inspect* a better strategy. In this dynamic process, the long-term averages of the fractions approximate the equilibrium probabilities.

6 Extensive games with perfect information

Games in strategic form have no temporal component. In a game in strategic form, the players choose their strategies simultaneously, without knowing the choices of the other players. The more detailed model of a *game tree*, also called a game in *extensive form*, formalizes interactions where the players can over time be informed about the actions of others. This section treats games of *perfect information*. In an extensive game with perfect information, every player is at any point aware of the previous choices of all other players. Furthermore, only one player moves at a time, so that there are no simultaneous moves.

Example: Quality choice with commitment

Figure 7 shows another variant of the quality choice game. This is a game tree with perfect information. Every branching point, or *node*, is associated with a player who makes a move by choosing the next node. The connecting lines are labeled with the player's choices. The game starts at the initial node, the *root* of the tree, and ends at a terminal node, which establishes the outcome and determines the players' payoffs. In Figure 7, the tree grows from left to right; game trees may also be drawn top-down or bottom-up.

The service provider, player I, makes the first move, choosing *High* or *Low* quality of service. Then the customer, player II, is informed about that choice. Player II can then decide separately between *buy* and *don't buy* in each case. The resulting payoffs are the

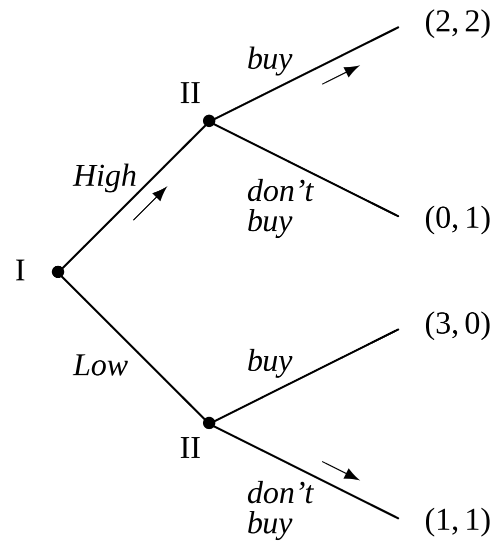

Figure 7. Quality choice game where player I *commits* to *High* or *Low* quality, and player II can react accordingly. The arrows indicate the optimal moves as determined by backward induction.

same as in the strategic-form game in Figure 3. However, the game is different from the one in Figure 3, since the players now move in sequence rather than simultaneously.

Extensive games with perfect information can be analyzed by *backward induction*. This technique solves the game by first considering the last possible choices in the game. Here, player II moves last. Since she knows the play will end after her move, she can safely select the action which is best for her. If player I has chosen to provide high-quality service, then the customer prefers to *buy*, since her resulting payoff of 2 is larger than 1 when not buying. If the provider has chosen *Low*, then the customer prefers not to purchase. These choices by player II are indicated by arrows in Figure 7.

Once the last moves have been decided, backward induction proceeds to the players making the next-to-last moves (and then continues in this manner). In Figure 7, player I makes the next-to-last move, which in this case is the first move in the game. Being rational, he anticipates the subsequent choices by the customer. He therefore realizes that his decision between *High* and *Low* is effectively between the outcomes with payoffs $(2,2)$ or $(1,1)$ for the two players, respectively. Clearly, he prefers *High*, which results in a payoff of 2 for him, to *Low*, which leads to an outcome with payoff 1. So the unique solution to the game, as determined by backward induction, is that player I offers high-quality service, and player II responds by buying the service.

Strategies in extensive games

In an extensive game with perfect information, backward induction usually prescribes unique choices at the players' decision nodes. The only exception is if a player is indifferent between two or more moves at a node. Then, any of these best moves, or even randomly selecting from among them, could be chosen by the analyst in the backward induction process. Since the eventual outcome depends on these choices, this may affect a player who moves earlier, since the anticipated payoffs of that player may depend on the subsequent moves of other players. In this case, backward induction does not yield a unique outcome; however, this can only occur when a player is exactly indifferent between two or more outcomes.

The backward induction solution specifies the way the game will be played. Starting from the root of the tree, play proceeds along a path to an outcome. Note that the analysis yields more than the choices along the path. Because backward induction looks at every node in the tree, it specifies for every player a *complete plan* of what to do at every point in the game where the player can make a move, even though that point may never arise in the course of play. Such a plan is called a *strategy* of the player. For example, a strategy of player II in Figure 7 is "buy if offered high-quality service, don't buy if offered low-quality service." This is player II's strategy obtained by backward induction. Only the first choice in this strategy comes into effect when the game is played according to the backward-induction solution.

I \ II	*H: buy, L: buy*	*H: buy, L: don't*	*H: don't, L: buy*	*H: don't, L: don't*
High	2 2	2 2	1 0	1 0
Low	0 3	1 1	0 3	1 1

Figure 8. Strategic form of the extensive game in Figure 7.

With strategies defined as complete move plans, one can obtain the *strategic form* of the extensive game. As in the strategic form games shown before, this tabulates all strategies of the players. In the game tree, any strategy combination results into an outcome of the game, which can be determined by tracing out the path of play arising from the players adopting the strategy combination. The payoffs to the players are then entered into the corresponding cell in the strategic form. Figure 8 shows the strategic form for our example. The second column is player II's backward induction strategy, where "buy if offered high-quality service, don't buy if offered low-quality service" is abbreviated as *H: buy, L: don't*.

A game tree can therefore be analyzed in terms of the strategic form. It is not hard to see that backward induction always defines a Nash equilibrium. In Figure 8, it is the strategy combination (*High; H: buy, L: don't*).

A game that evolves over time is better represented by a game tree than using the strategic form. The tree reflects the temporal aspect, and backward induction is succinct and natural. The strategic form typically contains redundancies. Figure 8, for example, has eight cells, but the game tree in Figure 7 has only four outcomes. Every outcome appears twice, which happens when two strategies of player II differ only in the move that is not reached after the move of player I. All move combinations of player II must be distinguished as strategies since any two of them may lead to different outcomes, depending on the action of player I.

Not all Nash equilibria in an extensive game arise by backward induction. In Figure 8, the rightmost bottom cell (*Low; H: don't, L: don't*) is also an equilibrium. Here the customer never buys, and correspondingly *Low* is the best response of the service provider to this anticipated behavior of player II. Although *H: don't* is not an optimal choice (so it disagrees with backward induction), player II never has to make that move, and is therefore not better off by changing her strategy. Hence, this is indeed an equilibrium. It prescribes a suboptimal move in the *subgame* where player II has learned that player I has chosen *High*. Because a Nash equilibrium obtained by backward induction does not have such a deficiency, it is also called *subgame perfect*.

The strategic form of a game tree may reveal Nash equilibria which are not subgame perfect. Then a player plans to behave irrationally in a subgame. He may even profit from this *threat* as long as he does not have to execute it (that is, the subgame stays unreached). Examples are games of *market entry deterrence*, for example the so-called "chain store" game. The analysis of dynamic strategic interaction was pioneered by Selten, for which he earned a share of the 1994 Nobel prize.

First-mover advantage

A practical application of game-theoretic analysis may be to reveal the potential effects of changing the "rules" of the game. This has been illustrated with three versions of the quality choice game, with the analysis resulting in three different predictions for how the game might be played by rational players. Changing the original quality choice game in Figure 3 to Figure 4 yielded an additional, although not unique, Nash equilibrium (*High, buy*). The change from Figure 3 to Figure 7 is more fundamental since there the provider has the power to *commit* himself to high or low quality service, and inform the customer of that choice. The backward induction equilibrium in that game is unique, and the outcome is better for both players than the original equilibrium (*Low, don't buy*).

Many games in strategic form exhibit what may be called the *first-mover advantage*. A player in a game becomes a first mover or "leader" when he can *commit* to a strategy, that is, choose a strategy irrevocably and inform the other players about it; this is a change of the "rules of the game." The first-mover advantage states that a player who can become a leader is not worse off than in the original game where the players act simultaneously. In other words, if one of the players has the power to commit, he or she should do so.

This statement must be interpreted carefully. For example, if more than one player has the power to commit, then it is not necessarily best to go first. For example, consider changing the game in Figure 3 so that player II can commit to her strategy, and player I moves second. Then player I will always respond by choosing *Low*, since this is his dominant choice in Figure 3. Backward induction would then amount to player II not buying and player I offering low service, with the low payoff 1 to both. Then player II is not worse off than in the simultaneous-choice game, as asserted by the first-mover advantage, but does not gain anything either. In contrast, making player I the first mover as in Figure 7 is beneficial to both.

If the game has antagonistic aspects, like the inspection game in Figure 6, then mixed strategies may be required to find a Nash equilibrium of the simultaneous-choice game. The first-mover game always has an equilibrium, by backward induction, but having to commit and inform the other player of a pure strategy may be disadvantageous. The correct comparison is to consider commitment to a *randomized choice*, like to a certain inspection probability. In Figure 6, already the commitment to the pure strategy *Inspect* gives a better payoff to player I than the original mixed equilibrium since player II will respond by complying, but a commitment to a sufficiently high inspection probability (anything above 10 percent) is even better for player I.

Example: Duopoly of chip manufacturers

The first-mover advantage is also known as *Stackelberg leadership*, after the economist Heinrich von Stackelberg who formulated this concept for the structure of markets in 1934. The classic application is to the duopoly model by Cournot, which dates back to 1838.

I \ II	*h*	*m*	*l*	*n*
H	0 0	8 12	9 18	0 36
M	12 8	16 16	15 20	0 32
L	18 9	20 15	18 18	0 27
N	36 0	32 0	27 0	0 0

Figure 9. Duopoly game between two chip manufacturers who can decide between high, medium, low, or no production, denoted by H, M, L, N for firm I and h, m, l, n for firm II. Prices fall with increased production. Payoffs denote profits in millions of dollars.

As an example, suppose that the market for a certain type of memory chip is dominated by two producers. The firms can choose to produce a certain quantity of chips, say either high, medium, low, or none at all, denoted by H, M, L, N for firm I and h, m, l, n for firm II. The market price of the memory chips decreases with increasing total quantity produced by both companies. In particular, if both choose a high quantity of production, the price collapses so that profits drop to zero. The firms know how increased production lowers the chip price and their profits. Figure 9 shows the game in strategic form, where both firms choose their output level simultaneously. The symmetric payoffs are derived from Cournot's model, explained below.

The game can be solved by dominance considerations. Clearly, no production is dominated by low or medium production, so that row N and column n in Figure 9 can be eliminated. Then, high production is dominated by medium production, so that row H and column h can be omitted. At this point, only medium and low production remain. Then, regardless of whether the opponent produces medium or low, it is always better for each firm to produce medium. Therefore, the Nash equilibrium of the game is (M, m), where both firms make a profit of \$16 million.

Consider now the commitment version of the game, with a game tree (omitted here) corresponding to Figure 9 just as Figure 7 is obtained from Figure 3. Suppose that firm I is able to publicly announce and commit to a level of production, given by a row in Figure 9. Then firm II, informed of the choice of firm I, will respond to H by l (with maximum payoff 9 to firm II), to M by m, to L also by m, and to N by h. This determines the backward induction strategy of firm II. Among these anticipated responses by firm II, firm I does best by announcing H, a high level of production. The backward induction outcome is thus that firm I makes a profit \$18 million, as opposed to only \$16 million in the simultaneous-choice game. When firm II must play the role of the follower, its profits fall from \$16 million to \$9 million.

The first-mover advantage again comes from the ability of firm I to credibly commit itself. After firm I has chosen H, and firm II replies with l, firm I would like to be able switch to M, improving profits even further from \$18 million to \$20 million. However, once firm I is producing M, firm II would change to m. This logic demonstrates why, when the firms choose their quantities simultaneously, the strategy combination (H, l) is not an equilibrium. The commitment power of firm I, and firm II's appreciation of this fact, is crucial.

The payoffs in Figure 9 are derived from the following simple model due to Cournot. The high, medium, low, and zero production numbers are 6, 4, 3, and 0 million memory chips, respectively. The profit per chip is $12 - Q$ dollars, where Q is the total quantity (in millions of chips) on the market. The entire production is sold. As an example, the strategy combination (H, l) yields $Q = 6 + 3 = 9$, with a profit of \$3 per chip. This yields the payoffs of 18 and 9 million dollars for firms I and II in the (H, l) cell in Figure 9. Another example is firm I acting as a monopolist (firm II choosing n), with a high production level H of 6 million chips sold at a profit of \$6 each.

In this model, a monopolist would produce a quantity of 6 million even if other numbers than 6, 4, 3, or 0 were allowed, which gives the maximum profit of \$36 million. The two firms could cooperate and split that amount by producing 3 million each, corresponding to the strategy combination (L, l) in Figure 9. The equilibrium quantities, however, are 4 million for each firm, where both firms receive less. The central four cells in Figure 9, with low and medium production in place of "cooperate" and "defect," have the structure of a Prisoner' Dilemma game (Figure 1), which arises here in a natural economic context. The optimal commitment of a first mover is to produce a quantity of 6 million, with the follower choosing 3 million. These numbers, and the equilibrium ("Cournot") quantity of 4 million, apply even when arbitrary quantities are allowed (see Gibbons, 1992).

7 Extensive games with imperfect information

Typically, players do not always have full access to all the information which is relevant to their choices. Extensive games with *imperfect information* model exactly which information is available to the players when they make a move. Modeling and evaluating strategic information precisely is one of the strengths of game theory. John Harsanyi's pioneering work in this area was recognized in the 1994 Nobel awards.

Consider the situation faced by a large software company after a small startup has announced deployment of a key new technology. The large company has a large research and development operation, and it is generally known that they have researchers working on a wide variety of innovations. However, only the large company knows for sure whether or not they have made any progress on a product similar to the startup's new technology. The startup believes that there is a 50 percent chance that the large company has developed the basis for a strong competing product. For brevity, when the large company has the ability to produce a strong competing product, the company will be referred to as having a "strong" position, as opposed to a "weak" one.

The large company, after the announcement, has two choices. It can counter by announcing that it too will release a competing product. Alternatively, it can choose to cede the market for this product. The large company will certainly condition its choice upon its private knowledge, and may choose to act differently when it has a strong position than when it has a weak one. If the large company has announced a product, the startup is faced with a choice: it can either negotiate a buyout and sell itself to the large company, or it can remain independent and launch its product. The startup does not have access to the large firm's private information on the status of its research. However, it does observe whether or not the large company announces its own product, and may attempt to infer from that choice the likelihood that the large company has made progress of their own.

When the large company does not have a strong product, the startup would prefer to stay in the market over selling out. When the large company does have a strong product, the opposite is true, and the startup is better off by selling out instead of staying in.

Figure 10 shows an extensive game that models this situation. From the perspective of the startup, whether or not the large company has done research in this area is random. To capture random events such as this formally in game trees, *chance moves* are introduced. At a node labelled as a chance move, the next branch of the tree is taken randomly and non-strategically by chance, or "nature", according to probabilities which are included in the specification of the game.

The game in Figure 10 starts with a chance move at the root. With equal probability 0.5, the chance move decides if the large software company, player I, is in a strong position (upward move) or weak position (downward move). When the company is in a weak position, it can choose to *Cede* the market to the startup, with payoffs $(0, 16)$ to the two players (with payoffs given in millions of dollars of profit). It can also *Announce* a competing product, in the hope that the startup company, player II, will *sell out*, with payoffs 12 and 4 to players I and II. However, if player II decides instead to *stay in*, it will even profit from the increased publicity and gain a payoff of 20, with a loss of -4 to the large firm.

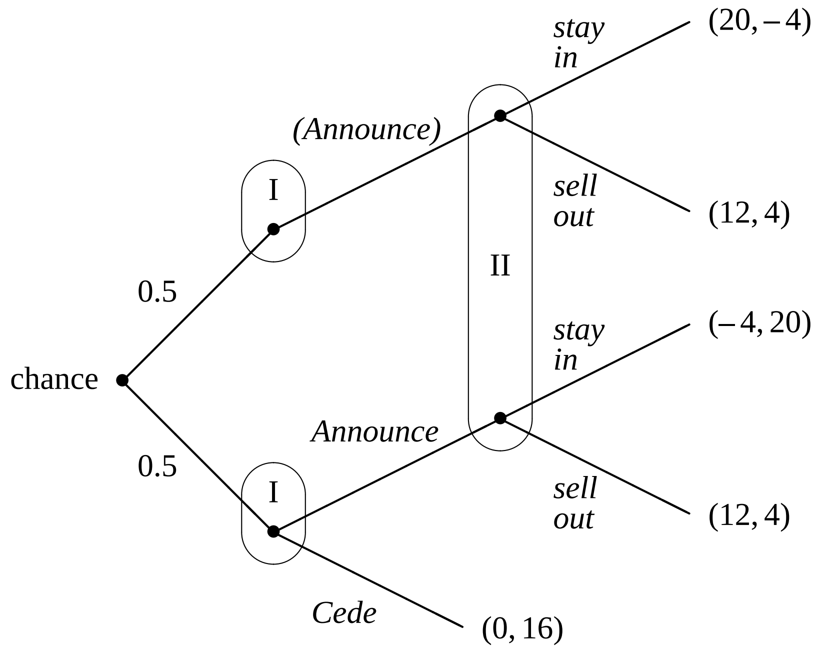

Figure 10. Extensive game with imperfect information between player I, a large software firm, and player II, a startup company. The chance move decides if player I is strong (top node) and does have a competing product, or weak (bottom node) and does not. The ovals indicate information sets. Player II sees only that player I chose to *Announce* a competing product, but does not know if player I is strong or weak.

In contrast, when the large firm is in a strong position, it will not even consider the move of ceding the market to the startup, but will instead just announce its own product. In Figure 10, this is modeled by a single choice of player I at the upper node, which is taken for granted (one could add the extra choice of ceding and subsequently eliminate it as a dominated choice of the large firm). Then the payoffs to the two players are $(20, -4)$ if the startup stays in and $(12, 4)$ if the startup sells out.

In addition to a game tree with perfect information as in Figure 7, the nodes of the players are enclosed by ovals which are called *information sets*. The interpretation is that a player cannot distinguish among the nodes in an information set, given his knowledge at the time he makes the move. Since his knowledge at all nodes in an information set is the same, he makes the same choice at each node in that set. Here, the startup company, player II, must choose between *stay in* and *sell out*. These are the two choices at player II's information set, which has two nodes according to the different histories of play, which player II cannot distinguish.

Because player II is not informed about its position in the game, backward induction can no longer be applied. It would be better to *sell out* at the top node, and to *stay in* at the bottom node. Consequently, player I's choice when being in the weak position is not clear: if player II stays in, then it is better to *Cede* (since 0 is better than -4), but if player II sells out, then it is better to *Announce*.

The game does not have an equilibrium in pure strategies: The startup would respond to *Cede* by selling out when seeing an announcement, since then this is only observed when player I is strong. But then player I would respond by announcing a product even in the weak position. In turn, the equal chance of facing a strong or weak opponent would induce the startup to stay in, since then the expected payoff of $0.5(-4) + 0.5 \times 20 = 8$ exceeds 4 when selling out.

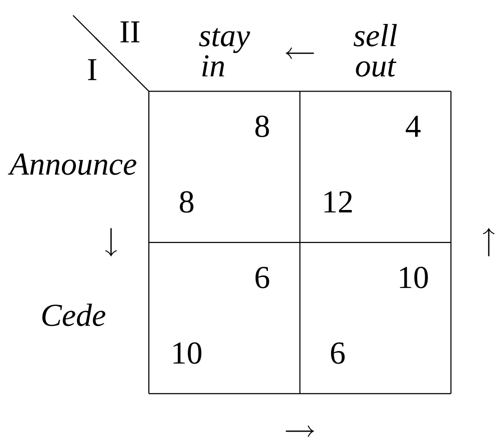

Figure 11. Strategic form of the extensive game in Figure 10, with expected payoffs resulting from the chance move and the player's choices.

The equilibrium of the game involves both players randomizing. The mixed strategy probabilities can be determined from the strategic form of the game in Figure 11. When it is in a weak position, the large firm randomizes with equal probability 1/2 between *Announce* and *Cede* so that the expected payoff to player II is then 7 for both *stay in* and *sell out*.

Since player II is indifferent, randomization is a best response. If the startup chooses to *stay in* with probability 3/4 and to *sell out* with probability 1/4, then player I, in turn, is indifferent, receiving an overall expected payoff of 9 in each case. This can also be seen from the extensive game in Figure 10: when in a weak position, player I is indifferent between the moves *Announce* and *Cede* where the expected payoff is 0 in each case. With probability 1/2, player I is in the strong position, and stands to gain an expected payoff of 18 when facing the mixed strategy of player II. The overall expected payoff to player I is 9.

8 Zero-sum games and computation

The extreme case of players with fully opposed interests is embodied in the class of two-player *zero-sum* (or constant-sum) games. Familiar examples range from rock-paper-scissors to many parlor games like chess, go, or checkers.

A classic case of a zero-sum game, which was considered in the early days of game theory by von Neumann, is the game of poker. The extensive game in Figure 10, and its strategic form in Figure 11, can be interpreted in terms of poker, where player I is dealt a strong or weak hand which is unknown to player II. It is a *constant-sum* game since for any outcome, the two payoffs add up to 16, so that one player's gain is the other player's loss. When player I chooses to announce despite being in a weak position, he is colloquially said to be "bluffing." This bluff not only induces player II to possibly sell out, but similarly allows for the possibility that player II stays in when player I is strong, increasing the gain to player I.

Mixed strategies are a natural device for constant-sum games with imperfect information. Leaving one's own actions open reduces one's vulnerability against malicious responses. In the poker game of Figure 10, it is too costly to bluff all the time, and better to randomize instead. The use of active randomization will be familiar to anyone who has played rock-paper-scissors.

Zero-sum games can be used to model strategically the computer science concept of "demonic" nondeterminism. Demonic nondeterminism is based on the assumption that, when an ordering of events is not specified, one must assume that the worst possible sequence will take place. This can be placed into the framework of zero-sum game theory by treating nature (or the environment) as an antagonistic opponent. Optimal randomization by such an opponent describes a worst-case scenario that can serve as a benchmark.

A similar use of randomization is known in the theory of algorithms as Rao's theorem, and describes the power of randomized algorithms. An example is the well-known *quick-sort algorithm*, which has one of the best observed running times of sorting algorithms in practice, but can have bad worst cases. With randomization, these can be made extremely unlikely.

Randomized algorithms and zero-sum games are used for analyzing problems in *online computation*. This is, despite its name, not related to the internet, but describes the situation where an algorithm receives its input one data item at a time, and has to make decisions, for example in scheduling, without being able to wait until the entirety of the input is known. The analysis of online algorithms has revealed insights into hard optimization problems, and seems also relevant to the massive data processing that is to be expected in the future. At present, it constitutes an active research area, although mostly confined to theoretical computer science (see Borodin and El-Yaniv, *Online Computation and Competitive Analysis*, Cambridge University Press, 1998).

9 Bidding in auctions

The design and analysis of auctions is one of the triumphs of game theory. Auction theory was pioneered by the economist William Vickrey in 1961. Its practical use became apparent in the 1990s, when auctions of radio frequency spectrum for mobile telecommunication raised billions of dollars. Economic theorists advised governments on the design of these auctions, and companies on how to bid (see McMillan, "Selling spectrum rights," *Journal of Economic Perspectives* Vol. 8, 1994, pages 145–162). The auctions for spectrum rights are complex. However, many principles for sound bidding can be illustrated by applying game-theoretic ideas to simple examples. This section highlights some of these examples; see Milgrom, "Auctions and bidding: a primer" (*Journal of Economic Perspectives* Vol. 3, 1989, pages 3–22) for a broader view of the theory of bidding in auctions.

Second-price auctions with private values

The most familiar type of auction is the familiar *open ascending-bid* auction, which is also called an *English* auction. In this auction format, an object is put up for sale. With the potential buyers present, an auctioneer raises the price for the object as long as two or more bidders are willing to pay that price. The auction stops when there is only one bidder left, who gets the object at the price at which the last remaining opponent drops out.

A complete analysis of the English auction as a game is complicated, as the extensive form of the auction is very large. The observation that the winning bidder in the English auction pays the amount at which the last remaining opponent drops out suggests a simpler auction format, the *second-price* auction, for analysis. In a second-price auction, each potential buyer privately submits, perhaps in a sealed envelope or over a secure computer connection, his bid for the object to the auctioneer. After receiving all the bids, the auctioneer then awards the object to the bidder with the highest bid, and charges him the amount of the second-highest bid. Vickrey's analysis dealt with auctions with these rules.

How should one bid in a second-price auction? Suppose that the object being auctioned is one where the bidders each have a *private value* for the object. That is, each bidder's value derives from his personal tastes for the object, and not from considerations such as potential resale value. Suppose this valuation is expressed in monetary terms, as the maximum amount the bidder would be willing to pay to buy the object. Then the optimal bidding strategy is to submit a bid equal to one's actual value for the object.

Bidding one's private value in a second-price auction is a *weakly dominant* strategy. That is, irrespective of what the other bidders are doing, no other strategy can yield a better outcome. (Recall that a dominant strategy is one that is *always* better than the dominated strategy; weak dominance allows for other strategies that are sometimes equally good.) To see this, suppose first that a bidder bids less than the object was worth to him. Then if he wins the auction, he still pays the second-highest bid, so nothing changes. However, he now risks that the object is sold to someone else at a lower price than his true valuation, which makes the bidder worse off. Similarly, if one bids more than one's value, the only case where this can make a difference is when there is, below the new bid, another bid exceeding the own value. The bidder, if he wins, must then pay that price, which he prefers less than not winning the object. In all other cases, the outcome is the same. Bidding one's true valuation is a simple strategy, and, being weakly dominant, does not require much thought about the actions of others.

While second-price sealed-bid auctions like the one described above are not very common, they provide insight into a Nash equilibrium of the English auction. There is a strategy in the English auction which is analogous to the weakly dominant strategy in the second price auction. In this strategy, a bidder remains active in the auction until the price exceeds the bidder's value, and then drops out. If all bidders adopt this strategy, no bidder can make himself better off by switching to a different one. Therefore, it is a Nash equilibrium when all bidders adopt this strategy.

Most online auction websites employ an auction which has features of both the English and second-price rules. In these auctions, the current price is generally observable to all participants. However, a bidder, instead of frequently checking the auction site for the current price, can instead instruct an agent, usually an automated agent provided by the auction site, to stay in until the price surpasses a given amount. If the current bid is by another bidder and below that amount, then the agent only bids up the price enough so that it has the new high bid. Operationally, this is similar to submitting a sealed bid in a second-price auction. Since the use of such agents helps to minimize the time investment needed for bidders, sites providing these agents encourage more bidders to participate, which improves the price sellers can get for their goods.

Example: Common values and the winner's curse

A crucial assumption in the previous example of bidding in a second-price auction is that of private values. In practice, this assumption may be a very poor approximation. An object of art may be bought as an investment, and a radio spectrum license is acquired for business reasons, where the value of the license depends on market forces, such as the demand for mobile telephone usage, which have a common impact on all bidders. Typically, auctions have both private and *common value* aspects.

In a purely common value scenario, where the object is worth the same to all bidders, bidders must decide how to take into account uncertainty about that value. In this case, each bidder may have, prior to the auction, received some private information or signals about the value of the object for sale. For example, in the case of radio spectrum licenses, each participating firm may have undertaken its own market research surveys to estimate the retail demand for the use of that bandwidth. Each survey will come back with slightly different results, and, ideally, each bidder would like to have access to all the surveys in formulating its bid. Since the information is proprietary, that is not possible.

Strategic thinking, then, requires the bidders to take into account the additional information obtained by winning the auction. Namely, the sheer fact of winning means that one's own, private information about the worth of the object was probably overly optimistic, perhaps because the market research surveys came back with estimates for bandwidth demand which were too bullish. Even if everybody's estimate about that worth is correct on average, the largest (or smallest) of these estimates is not. In a procurement situation, for example, an experienced bidder should add to his own bid not only a markup for profit, but also for the likely under-estimation of the cost that results from the competitive selection process. The principle that winning a common-value auction is "bad news" for the winner concerning the valuation of the object is called the *winner's curse*.

The following final example, whose structure was first proposed by Max Bazerman and William Samuelson, demonstrates the considerations underlying the winner's curse not for an auction, but in a simpler situation where the additional information of "winning" is crucial for the expected utility of the outcome. Consider a potential buyer who is preparing a final, "take it or leave it" offer to buy out a dot-com company. Because of potential synergies, both the buyer and the seller know that the assets of the dot-com are worth 50 percent more to the buyer than to the current owner of the firm. If the value of the company were publicly known, the parties could work out a profitable trade, negotiating a price where both would profit from the transaction.

However, the buyer does not know the exact value of the company. She believes that it is equally likely to be any value between zero and ten million dollars. The dot-com's current owners know exactly the value of retaining the company, because they have complete information on their company's operations. In this case, the expected value of the company to the current owners is five million dollars, and the expected value of the company to the prospective buyer is seven and a half million dollars. Moreover, no matter what the value of the company truly is, the company is always worth more to the buyer than it is to the current owner. With this in mind, what offer should the buyer tender to the dot-com as her last, best offer, to be accepted or rejected?

To find the equilibrium of this game, note that the current owners of the dot-com will accept any offer that is greater than the value of the company to them, and reject any offer that is less. So, if the buyer tenders an offer of five million dollars, then the dot-com owners will accept if their value is between zero and five million. The buyer, being strategic, then realizes that this implies the value of the company to her is equally likely to be anywhere between zero and seven and a half million. This means that, if she offers five million, the average value of the company, conditioning upon the owners of the dot-com accepting the offer, is only three and three-quarters million – less than the value of the offer. Therefore, the buyer concludes that offering five million will lead to an expected loss.

The preceding analysis does not depend on the amount of the offer. The buyer soon realizes that, no matter what offer she makes, when she takes into account the fact that the offer will be accepted only when the value of the dot-com turns out to be on the low end. The expected value of the company to the buyer, conditional on her offer being accepted, is always less than her offer. It is this updating of the buyer's beliefs, shifting her estimation of the dot-com's value to the low end, which embodies the winner's curse in this example. Having her offer accepted is bad news for the buyer, because she realizes it implies the value of the dot-com is low. The equilibrium in this game involves the buyer making an offer of zero, and the offer never being accepted.

This example is particularly extreme, in that no transaction is made even though everyone involved realizes that a transaction would be profitable to both sides. As is generally the case with noncooperative game theory, the equilibrium does depend on the details of the rules of the game, in this case, the assumption that one last, best offer is being made, which will either be accepted or rejected. In general, the winner's curse will not always prohibit mutually profitable transactions from occurring. This example demonstrates the importance of carefully taking into account the information one learns during the course of play of a game. It also shows how a game-theoretic model that incorporates the information and incentives of others helps promote sound decision-making.

10 Further reading

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